On the Ross-Littlewood Paradox

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1 Description of the Paradox

Suppose that you have a computer with an infinite storage capacity, in other words, no matter how many characters you type, you will never run out of storage.

One day, starting at 11:00 a.m. you type 10 "a"s (due to boredom in a non-STEM class) in a row. You do this instantaneously, violating the speed of light and tearing apart the fabric of the universe, but just as instantaneously, God fixes the rift you tore in spacetime, and punishes your destructive action by removing one of the "a"s that you typed. However, you are a habitual sinner by nature, and after a half hour passes you do the same thing, and God, always being fair and merciful, does the same thing he did the first time. You provoke fate and morality $\frac{1}{3}$ hour later with the same results and so on until noon when the class ends and you decided to check what you have typed. How many "a"s did you type? And will you go to hell for your insolence?

The second question is not really answerable by anyone but God so we will give him the opportunity to answer that:

Ok. Assuming he decided to miraculously answer the second question, let us proceed with the analysis of the first question.

There are generally three schools of thought. Some mathematicians say that there are an infinite number of "a"s by the following argument. There are effectively 9 "a"s typed at each discrete moment so if you allow the number of moments $n \to \infty$ then $\lim_{n\to\infty} 9n$ diverges to ∞. So they say that there are a countably infinite amount of "a"s in the storage of the computer by noon. On the other hand since there are an infinite number of steps to the process, and at every step an "a" is backspaced out of existence by God you have that a countably infinite number of "a"s have been removed, thus canceling out the first infinity. The third response is that the problem has not been well defined and so cannot reasonably be answered, and they attempt to define it in specific

ways that tend to lead to the infinite or empty answer. Thus it is a paradox, it is all or nothing and both at the same time. Unless you can prove otherwise.
I should mention that obviously there are many equivalent ways of restating the paradox as I defined it. Mathematicians don't tend to mention the wrath of God in their descriptions of paradoxes.

1. "What is the result of a computer program computing an infinite number of steps in finite time?" $\iff$
2. "If I add 9 balls to an urn and remove 1 ball each step at time intervals of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ how many balls do I have in the urn by the completion of the infinite series?" [1] $\iff$
3. Building off the previous example, Ross made a probabilistic contribution (lending his name to the title of the paradox) and asks "if I remove one of the balls uniformly at each step what is the expected number of balls at the end of the process?". $\iff$

And so on. The essence is that a supertask (an infinite series of operations conducted in a finite time) leads to difficulties in identifying the overall result of the task.

2 Historical Background

This paradox was originally an analogue of Zeno's Paradox which essentially removes the operation at every step and the measurement at the end. In Zeno's Paradox, a man has a distance to travel of length n (say a mile, which many of students cross in a day of wandering between classes) and he first is required to travel half of the length. He then must travel half of the remaining length, and so on, traveling progressive halves. However, it seems like he will never traverse the whole mile by doing so since he is only traveling sums of lengths that are not equal to the total mile at any given point.

In his book, A Mathematician's Miscellany, John Littlewood originally presented a version of what we have described in the previous section in 1953, with urns and balls [2] building off of Zeno. Sheldon Ross (as previously mentioned) further added the uniform selection of the balls in his 1988 A First Course in Probability [3] which seeks to reduce the dependence on the order of the balls that are removed which had been used by philosophers to argue for the ability to vary the amount left in the container/storage at the end of the finite interval.

3 Attempts to Resolve the Paradox

3.1 The Problem Does Not Make Sense

The problem may be under-specified in the sense that what happens at noon (or whatever the end of the finite interval is) is not described in the problem. The operation that we take to observe the number of balls is not described so we cannot proceed to know what the correct answer to the problem is in any rigorous way. Moreover, it may be the case that it is impossible to tell anything about the qualities of the completion of the task from the infinite steps themselves, regardless of how they are characterized at a discrete time interval [4].

3.2 Continuity Makes it Easier

By assuming that the position of the objects that are not removed is a continuous function of time, and therefore an object that is not removed remains in its place at noon, we are able to more easily arrive at the conclusion that there is an infinite number of the objects within the container at the end of the finite interval. Another continuity assumption that is often made to come to the same conclusion is that at a time during which no balls are removed or added, the number of objects in the container is continuous, meaning that if you take the limit as t' approaches that time of continuity the number of balls at that time would find its limit. In other words, $\lim_{t' \to t} N(t') = N(t)$.[1]. It is supposed that these two versions of continuity (one in the number function and the other in the position function) must logically be related and since both seem reasonable one could conclude that the number at the end of the interval is infinite.

3.3 Reordering the Series

Let us examine a few possible reordering of the series described in paradox. First

$$1_0 + 1_1 + ... + 1_{10} - 1_0 + 1_{11} + 1_{12} + \dots \tag{1}$$

may be reordered as

$$\begin{aligned} &(1_0 - 1_0) + (1_1 - 1_1) + \dots \\ =&(0) + (0) + ... \\ =&0 \end{aligned}$$

However if you delayed the addition to the end you would have a countably infinite amount subtracting from a countably infinite amount which would yield a countably infinite amount, since you could remove 1 from the infinite number an infinite number of times without changing the size of the sum (if you could remove 1 a finite number of times and yield zero this would imply that the sum was not finite but whichever n for the number of times you removed 1).

$$1_0 + \cdots + 1_n + ... - (1_0 + 1_1 + \dots) \neq 0$$

This all depends on the absolute or conditional convergences of the sum, but as written in (1) the series diverges. But this series is computed, we know in a finite length of time. So what can a divergent series be said to converge to? It seems unfathomable that we can come to an answer. Even if we prove convergence with some strange assumption, there would be the difficulty of determining whether it is convergent absolutely or conditionally (in which case, by Riemann, one could rearrange the terms to make it equal to any sum). Some philosophers and logicians argue that reordering could yield results but I am unconvinced as of yet.

4 The Upshot for Computation

It is clear that the paradox does not necessarily yield an easy answer. However, there have been some interesting models of computing that have been developed as a result of the study of this problem. For example, the accelerated Turing machine is a direct analogue of this problem which can be described in the following way. An accelerated Turing machine takes 2^n minutes to complete an operation where n is the number of steps since the first operation including that operation. Starting with $n = 0$ we may wait 2 minutes and ask a question about the result of the Turing machine's process, its supertask. In one example, the operation was Succ(x) where x started at 0 and the question was "is the final number at the end of 2 minutes even or odd?"[5]. This computer obviously could never be built in physical reality (it does violate the speed of light after all which has been shown to be constant through the universe). Indeed it is unclear whether there is a logical answer that can arise from its operations, for the reasons we described earlier.

The questions we can ask about are interesting, however. For example, does it solve the halting problem, thus increasing the computational power over the classical Turing machine? The answer to this question is debated but has been argued to be yes (though this has been harshly challenged by other thinkers) [5]. The argument goes in the following way: "A classical Turing machine is simulated on an accelerated Turing machine: if it halts then a symbol 1 is printed on the tape after the operation, if not 0 is printed on the tape, and all the operations of the classical Turing machine are done by the accelerated Turing machine in the 2 (or so) minutes allotted to complete an infinite series of operations. Thus the machine decides the halting problem." This would certainly tend towards the conclusion that the ATM has greater computational power over the classical Turing machine. However, I think that the argument made by another thinker also holds merit that this assumes we are able to study the end behavior which implies some kind of observation action at the end of the two minutes that needs to be specified mathematically. Ultimately, it is difficult to judge which is correct and I will not attempt to do so here.

References

[1] John Byl. "On resolving the Littlewood-Ross paradox". In: *Missouri Journal of Mathematical Sciences* 12.1 (2000), pp. 42–47.

[2] John Edensor Littlewood. *A Mathematician's Miscellany.* London: Methuen, 1953.

[3] Sheldon M. Ross. *A First Course in Probability.* 3rd. New York: Macmillan, 1988.

[4] John Earman and John D. Norton. "Infinite Pains: The Trouble with Supertasks". In: *Benacerraf and His Critics.* Ed. by Adam Morton and Stephen P. Stich. Oxford; Cambridge, MA: Blackwell, 1996, pp. 231–261.

[5] Gentian Kasa. *Hypercomputation: Towards an Extension of the Classical Notion of Computability?* 2012. arXiv: `1210.7171 [cs.LO]`. URL: `https://arxiv.org/abs/1210.7171`.